Ward identities for disordered metals and superconductors

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Abstract

This article revisits Ward identities for disordered interacting normal metals and superconductors. It offers a simple derivation based on gauge invariance and recasts the identities in a new form that allows easy analysis of the quasi-particle charge conservation (as e.g. in a normal metal) or non-conservation (as e.g. in a d-wave superconductor).

I. INTRODUCTION

Interplay of interaction and disorder remains one of the central topics in condensed matter physics. Given the complexity of the problem, constraints imposed by symmetries acquire particular importance. An example of such a constraint is given by Ward identities. In the early days of many-body theory, Ward identities were used to establish key properties of the Fermi liquid theory.[1] In the context of the CPA approximation for a disordered non-interacting metal, similar identities were derived by Velický.[2] In the theory of superconductivity, Ward identities were used early on to establish gauge invariance of the electromagnetic response.[3] Subsequently, they were employed by D. Vollhardt and P. Wölfle[4] in a self-consistent theory of the Anderson transition, by F. Wegner,[5] A. J. McKane and M. Stone[6] in the sigma model approach to localization, and by C. Castellani *et al.*[7] in an early treatment of an interacting disordered metal. Very recently, T. R. Kirkpatrick and D. Belitz[8] invoked the Ward identities in an attempt to resolve the issue of decoherence at zero temperature.

This paper revisits the Ward identities for disordered interacting normal metals and superconductors. Using gauge invariance, it derives the identities in a new form that makes quasiparticle charge conservation (as e.g. in a normal metal) or absence thereof (as e.g. in a d-wave superconductor) explicit. In a normal metal, the identity takes a particularly simple form:

$$\Lambda_{RA}(\omega,\omega';p,p) = -\frac{2i\Sigma''_R(\omega,p)}{\omega-\omega'},$$

where Λ_{RA} is the disorder average of the retarded-advanced charge density vertex correction at zero momentum transfer $Q = p - p = 0$ and small frequency transfer $\Omega = \omega - \omega' \ll \omega, \omega'$, and $\Sigma''_R(\omega,p)$ is the imaginary part of the retarded quasiparticle self energy, which is proportional to the quasiparticle scattering rate. The vertex Λ_{RA} is closely related to the correlation function of the quasiparticle charge density, and the $1/(\omega-\omega')$ behavior of the vertex at low frequency transfer and zero momentum transfer points to quasiparticle charge conservation and its diffusive propagation. By contrast, in a d-wave superconductor, the Ward identity reflects the fact that impurity scattering causes exchange of charge between the quasiparticle subsystem and the condensate, which leads to non-conservation of the quasiparticle charge.

The structure of the paper is as follows. Section II gives a detailed derivation of the Ward identities for a disordered interacting normal metal. Section III briefly discusses the Ward identities for an s-wave superconductor in the approximation of a spatially uniform gap. Section IV derives the Ward identifies for a disordered d-wave superconductor in the same approximation, and Section V illustrates the meaning of the identity by explaining how, in a d-wave superconductor, the impurity scattering leads to exchange of charge between the quasiparticle subsystem and the condensate. Section VI presents a summary and a brief discussion of the results.

II. WARD IDENTITIES FOR A NORMAL METAL

Consider a disordered interacting normal metal with the Matsubara action

$$S = \int d\tau \int dr \psi^+(r,\tau) \left[i\hbar\partial_\tau - \xi(-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}) + e\phi(r,\tau) - u(r) \right] \psi(r,\tau) -$$
$$- \int d\tau dr \int d\tau' dr' \psi^+_\alpha(r,\tau)\psi_\beta(r,\tau)V_{\alpha\beta\gamma\delta}(\tau-\tau', r-r')\psi^+_\gamma(r',\tau')\psi_\delta(r',\tau'),$$

where ψ^+ (ψ) are the electron creation (annihilation) operators, $\vec{A}$ is the electromagnetic vector potential, $\phi(r,\tau)$ is the scalar potential and $u(r)$ is the impurity potential. This action respects the continuous gauge symmetry

$$\psi \to e^{i\chi(r,\tau)}\psi; \quad \vec{A} \to \vec{A} + \frac{\hbar c}{e}\vec{\nabla}\chi; \quad \phi \to \phi + \frac{\hbar}{e}\partial_\tau\chi,$$

of which the sought Ward identities are a consequence. To establish the scheme used throughout the rest of this article, I present below a detailed derivation.

Everywhere hereafter, only infinitesimal time dependent spatially uniform transformations $\psi_\alpha(r,\tau) \to e^{i\chi(\tau)}\psi_\alpha(r,\tau)$ will be considered. Under such a transformation, the Green function changes according to

$$G_{\alpha\beta}(r,r',\tau-\tau') \to e^{i\chi(\tau)}G_{\alpha\beta}(r,r',\tau-\tau')e^{-i\chi(\tau')}$$

and thus, to first order in χ, its variation equals

$$\delta G_{\alpha\beta}(r,r',\tau-\tau') \approx i[\chi(\tau) - \chi(\tau')]G_{\alpha\beta}(r,r',\tau-\tau').$$

On the other hand, the same transformation induces extra terms in the action due to the presence of the temporal derivative. Hence, the very same variation of the Green function can also be calculated by perturbation theory. The crucial point is that the four-fermion interaction term in the action is invariant under the gauge transformation and, therefore, does not contribute to the perturbative correction to the Green function. Thus, to first order in infinitesimal χ, the same correction to G is equal to

$$\delta G_{\alpha\beta}(x,x',\tau-\tau') = -i \int dt dr \langle \psi_\alpha(x,\tau)\psi^+_\gamma(r,t)\psi_\gamma(r,t)\psi^+_\beta(x',\tau')\rangle \partial_t\chi(t).$$

Equating the two expressions leads to the identity

$$[\chi(\tau) - \chi(\tau')]G_{\alpha\beta}(x,x',\tau-\tau') = -\int dt dr \langle \psi_\alpha(x,\tau)\psi^+_\gamma(r,t)\psi_\gamma(r,t)\psi^+_\beta(x',\tau')\rangle \partial_t\chi(t)$$

for a given disorder configuration. Disorder averaging replaces the exact Green function on the left hand side by its translationally invariant average. The average on the right hand side can be presented as the product of the two average Green functions plus the vertex correction term and, for $\chi = \chi_0 e^{i\Omega\tau}$ with $\Omega \to 0$, the Fourier transformed identity takes the form

$$G(i\omega + i\Omega, p) - G(i\omega, p) = i\Omega G(i\omega + i\Omega, p)\,[1 + \Lambda(i\omega, i\omega + i\Omega; p, p)]\, G(i\omega, p),$$

where $\Lambda(i\omega, i\omega + i\Omega; p, p)$ is the disorder average of the scalar vertex correction. At this point, two different types of identities can be derived: one for the retarded-advanced vertex correction $\Lambda_{RA}(\omega, \omega + \Omega; p, p)$, and another one for the retarded-retarded vertex correction $\Lambda_{RR}(\omega, \omega + \Omega; p, p)$.

A. The identity for the retarded-advanced (RA) vertex

To obtain the identities for the retarded-advanced vertex, choose $i\omega$ to be in the lower half-plane and $i\omega+i\Omega$ in the upper half-plane. Then, upon analytic continuation $i\omega \to \omega \pm i0$, $G(i\omega)$ transforms into $G_A(\omega - i0)$, whereas $G(i\omega + i\Omega)$ transforms into $G_R(\omega + \Omega + i0)$. The identity then takes the form

$$G_R^{-1}(\omega + \Omega + i0, p) - G_A^{-1}(\omega - i0, p) = \Omega[1 + \Lambda_{RA}(\omega + \Omega, \omega; p, p)].$$

The disorder averaged Green function reads $G^{-1}_{A/R}(\omega, p) = \omega - \Sigma_{A/R}(\omega, p) - \xi(p)$, where $\Sigma_{A/R}(\omega, p)$ is the advanced/retarded self energy. Using the relation $\Sigma_R(\omega, p) = \Sigma_A^*(\omega, p)$ and assuming that the derivative $\partial_\omega \Sigma_{R/A}(\omega, p)$ is non-singular, for small $\Omega = \omega - \omega' \to 0$ one finds

$$-2i\Sigma_R''(\omega, p) = [\omega - \omega']\Lambda_{RA}(\omega, \omega'; p, p). \tag{1}$$

Identifying $2\Sigma_R''(\omega)$ with the scattering rate $1/\tau$, one immediately recognizes in Λ_{RA} the zero momentum transfer ($Q = 0$) limit of the charge density vertex $D(\omega - \omega', Q)$[9]

$$D(\omega-\omega',Q)=\frac{1}{i(\omega-\omega')\tau+DQ^2\tau}.$$

where D is the diffusion coefficient. For a non-interacting disordered metal, $D(\omega-\omega',Q)$ is commonly obtained by a direct calculation,[9] first finding self-consistently the impurity self energy, and then summing the ladder series for the vertex. In the presence of interactions, diagrammatic treatment becomes much more involved, while the present derivation appeals only to gauge invariance and is insensitive to turning on the interaction.

B. The identity for the retarded-retarded (RR) vertex

By contrast with the identity just derived, the identity for the retarded-retarded vertex can be found in textbooks, and I present its derivation here only for completeness. In this case, it is convenient to choose both $i\omega$ and $i\omega+i\Omega$ in the same (say, the upper) half-plane. Upon analytic continuation and multiplication by $G^{-1}(i\omega)$ and $G^{-1}(i\omega+i\Omega)$, the identity takes the form

$$G_R^{-1}(\omega+\Omega+i0,p)-G_R^{-1}(\omega+i0,p)=\Omega[1+\Lambda_{RR}(\omega+\Omega,\omega;p,p)],$$

which, to first order in $\Omega\to 0$, leads to the standard relation between the energy derivative of the retarded self energy and the retarded-retarded vertex[10] :

$$\partial_\omega\Sigma_R(\omega)=\Lambda_{RR}(\omega,\omega;p,p). \tag{2}$$

III. WARD IDENTITY FOR AN S-WAVE SUPERCONDUCTOR

In the Nambu notations, the BCS Hamiltonian of an s-wave superconductor reads

$$H=\int dr\Psi^\dagger\left[\tau_3\xi(\vec{p}-\frac{e}{c}\vec{A}\tau_3)+\tau_1\Delta(r)+\tau_3 e\phi+\tau_3 u\right]\Psi.$$

Here $\Psi^\dagger\equiv(\psi_\uparrow^\dagger,\psi_\downarrow)$ is the Nambu spinor, τ_i are the Nambu matrices, and r denotes the center of mass coordinate of a Cooper pair. The pair field $\Delta(r)$ has been chosen real for the sake of simplicity.

A gauge transformation takes the form $\Psi \to \exp\left[i\tau_3 \frac{e}{\hbar c}\chi\right]\Psi$ and, in addition to the standard change of potentials $\vec{A}$ and ϕ, has to be accompanied by the pair field transformation $\Delta \to \Delta \exp\left[2i\frac{e}{\hbar c}\chi\right]$. One then proceeds the same way as for a normal metal, with two important points to note. The first point amounts to the approximation of a spatially uniform gap which, along with the frequency, gets renormalized by disorder. The second point stems from the Nambu matrix structure of the theory: the vertices, that appear after disorder averaging of the perturbative expression for the Green function variation, are defined in the Nambu space and thus carry a Nambu index. For instance, the disorder averaged term arising from the temporal derivative of $\chi = \chi_0 \exp[i\Omega\tau]$ has the form

$$\int dy \langle \psi(x)\bar{\psi}(y) i\Omega\tau_3 \psi(y)\bar{\psi}(x')\rangle \to \int dydzdz' G(x,z) i\Omega \left[\tau_3 \delta(z-y)\delta(y-z') + \langle\tau_3\rangle(z,y,z')\right] G(z',x'),$$

where the vertex correction $\langle\tau_3\rangle(z,y,z')$ appears as a result of disorder dressing of the corresponding bare vertex, the latter being simply the Nambu matrix τ_3. The disorder averaged Green function has the form[1]

$$G(p,\omega) = [i\tilde{\omega}\tau_0 - \xi(p)\tau_3 - \tilde{\Delta}\tau_1]^{-1},$$

where $\tilde{\omega}$ and $\tilde{\Delta}$ are the renormalized frequency and the gap amplitude, which yields the Ward identity

$$[i\tilde{\omega} - i\tilde{\omega}']\,\tau_3 - i\left[\tilde{\Delta}_\omega + \tilde{\Delta}_{\omega'}\right]\tau_2 = [i\omega - i\omega'][\tau_3 + \langle\tau_3\rangle] - 2i\Delta[\tau_2 + \langle\tau_2\rangle].$$

This, in turn, indicates a diffusion pole in the quasiparticle charge density vertex correction $\langle\tau_3\rangle_{RA}$:

$$\frac{-2i\Sigma''_R(\omega,p)}{\omega-\omega'}\tau_3 = \langle\tau_3\rangle_{RA},$$

where $\Sigma''_R(\omega,p)$ is the imaginary part of the retarded self energy renormalization of the frequency, the notation is chosen to coincide with the normal metal limit.

IV. WARD IDENTITY FOR A D-WAVE SUPERCONDUCTOR:

In a d-wave superconductor, the situation turns out to be quite different. The BCS Hamiltonian of a d-wave superconductor reads

$$H = \int dr \Psi^{\dagger} \left[\tau_3 \xi(\vec{p} - \frac{e}{c}\vec{A}\tau_3) + \tau_3 e\phi + \tau_3 u \right] \Psi + \int dR dr \Psi^{\dagger}(R + \frac{r}{2}) \tau_1 \Delta(R, r) \Psi(R - \frac{r}{2}),$$

where the pair field $\Delta(R, r)$ has been chosen real and having d-wave angular dependence on the relative coordinate r, and R denotes the center of mass coordinate of a Cooper pair.

As in the s-wave case, the Hamiltonian respects the gauge symmetry, and the identities can be obtained similarly, with one important difference: because of the d-wave symmetry of the gap and its oscillating angular dependence, the gap amplitude Δ_p, although suppressed by impurities, does not acquire a frequency dependent renormalization. Hence the disorder average of the quasiparticle Green function is

$$G(i\omega, p) = \left[i\tilde{\omega} - \tau_1 \Delta_p - \tau_3 \xi_p \right]^{-1}.$$

Another important point is that the angular dependence of the gap leads to the appearance of a vertex correction $\langle \Delta_p \tau_2 \rangle$ on the right hand side of the Ward identity, which assumes the form

$$-2i\tau_3 \Sigma''_R(\omega, p) = [\omega - \omega'] \langle \tau_3 \rangle_{RA} + 2i \langle \Delta_p \tau_2 \rangle_{RA}. \tag{3}$$

As for an s-wave superconductor, $\Sigma''_R(\omega, p)$ is the retarded self-energy renormalization of the frequency: $\tilde{\omega} = \omega - \Sigma$. Due to the d-wave symmetry of the gap and its oscillatory angular dependence, $\langle \Delta_p \tau_2 \rangle_{RA} \propto \langle \tau_3 \rangle_{RA}$, which leads one to conclude that the vertex correction $\langle \tau_3 \rangle_{RA}$ has to remain finite as $\omega - \omega' \to 0$. Hence, in a disordered d-wave superconductor, the quasiparticle charge is not conserved. Note that, upon transition to the normal state, the quasiparticle charge diffusion mode re-appears, as can be seen seen by sending Δ_p to zero in Eq. (3) and identifying $\langle \tau_3 \rangle_{RA}$ with $\Lambda_{RA}(\omega, \omega'; p, p)$ of Section II.

V. QUALITATIVE ARGUMENT FOR A D-WAVE SUPERCONDUCTOR

The absence of quasiparticle charge conservation in a d-wave superconductor can be understood based on a simple argument going back to the studies of charge imbalance relaxation in superconductors.[11] I reproduce the argument here for the sake of completeness. Consider the Bogolyubov quasiparticle creation operator:

$$\gamma^{+}_{p\uparrow} = u_p c^{+}_{p\uparrow} + v_p c_{-p\downarrow} \ , \ u_p^2 = \frac{1}{2}[1 + \frac{\xi_p}{\sqrt{\xi_p^2 + \Delta_p^2}}] \ , \ u_p^2 + v_p^2 = 1.$$

Impurity scattering is elastic, i.e. it conserves the quasiparticle energy $E_p = \sqrt{\xi_p^2 + \Delta_p^2}$. In an s-wave superconductor with uniform gap, Δ_p is a constant and, in the absence of the Andreev scattering that turns ξ_p into $-\xi_p$, the energy conservation implies conservation of u_p and v_p . Hence the impurity scattering conserves the particle-hole content of a quasiparticle, and this leads to the effective charge conservation – even though a Bogolyubov quasiparticle, being a superposition of a particle and a hole, does not have a well defined charge quantum number. The same conclusion can be reached by considering directly the expectation value of the quasiparticle charge Q_p :

$$Q_p = u_p^2(+1) + v_p^2(-1) = \frac{\xi_p}{\sqrt{\xi_p^2 + \Delta_p^2}}.$$

In an isotropic s-wave superconductor, the gap does not vary around the Fermi surface and hence, in the absence of the Andreev processes, Q_p is conserved by the impurity scattering, which leads to the charge diffusion pole.

By contrast, in a d-wave superconductor, the gap Δ_p is strongly anisotropic. Thus, even in the absence of the Andreev scattering processes, neither Q_p nor the moduli of the Bogolyubov factors u_p and v_p are conserved: impurity scattering changes the particle-hole content of a quasiparticle. Physically, this means that the impurity scattering induces exchange of charge between the quasiparticle subsystem and the condensate.

Indeed, this quasiparticle charge non-conservation is not a consequence of the d-wave symmetry of the gap, but rather of the gap anisotropy around the Fermi surface, and is

present not only in other superconductors with non-trivial symmetry, but even in s-wave superconductors with anisotropic gap. However, in the latter case, the effect is small in the measure of the relative gap anisotropy, which is itself reduced by disorder. As a result, the quasiparticle charge non-conservation appears only at time scales that are long compared with the scattering time. By contrast, in a d-wave superconductor, the gap anisotropy is large, and the quasiparticle charge changes at the time scale of order the impurity scattering time, which eliminates quasiparticle charge conservation at any time scale beyond the elastic scattering time.

VI. SUMMARY AND DISCUSSION

In this article, I revisited the Ward identities for superconductors and disordered interacting normal metals, and presented a simple derivation based solely on gauge invariance. The identities were recast in a new form that made quasiparticle charge conservation (as in a normal metal or an isotropic s-wave superconductor) or absence thereof (as in a d-wave superconductor) explicit. Using the Ward identities, I showed how, in a d-wave superconductor, impurity scattering causes exchange of charge between the quasiparticle subsystem and the condensate, thus leading to the quasiparticle charge non-conservation.

Transparency of the Ward identities is particularly appealing in comparison with microscopic approaches. The simplicity of the identities is insensitive to the strength of the impurity potential or to whether disorder has to be treated in the Born or in the unitary limit – or to the presence of interaction. By contrast, to achieve a controllable approximation even in the Born limit, microscopic calculations, e.g. for a d-wave superconductor, have to resort to rather complex methods and/or unrealistic approximations, such as expansion in the inverse number of gap nodes.[12]

It is my pleasure to thank A. J. Leggett, M. R. Norman and especially Yu. M. Galperin for helpful discussions. Work supported by the U. S. Department of Energy, Division of Basic Energy Science-Material Science under contract No. W-31-109-ENG-38.

REFERENCES

[1] A. A. Abrikosov, L. P. Gorkov and I. E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics*, (Dover, 1975); P. Nozières, *Theory of Interacting Fermi Systems*, (Addison-Wesley, 1997).

[2] B. Velický, Phys. Rev. **184**, 614 (1969).

[3] See, e. g., J. R. Schrieffer, *Theory of Superconductivity*, Chapter 8. (Addison-Wesley, 1988); Y. Nambu, Phys. Rev. **117**, 648 (1960).

[4] D. Vollhardt and P. Wölfle, Phys. Rev. **B 22**, 4666 (1980).

[5] F. Wegner, Z. Phys. **B 35**, 207 (1979); L. Schfer and F. Wegner, *ibid.* B **38**, 113 (1980).

[6] A. J. McKane and M. Stone, Ann. Phys. **131**, 36, (1981).

[7] C. Castellani, C. Di Castro, G. Forgacs, E. Tabet, Nuclear Physics **B225**, 441 (1983).

[8] T. R. Kirkpatrick and D. Belitz, preprint cond-mat/0111398 and references therein.

[9] B. L. Altshuler and B. D. Simons, in *Mesoscopic Quantum Physics*, eds. E. Akkermans, G. Montambaux, J. -L. Pichard and J. Zinn-Justin (North Holland, Amsterdam, 1995).

[10] Gerald D. Mahan, *Many-Particle Physics*, Ch. 7 (Plenum, 1990).

[11] A lucid explanation can be found in V. V. Schmidt, *The Physics of Superconductors*, Chapter 7, (Springer-Verlag, 1997); for the original discussion, see M. Tinkham, Phys. Rev. **B 6**, 1747 (1972); M. Tinkham, *Introduction to Superconductivity*, Chapter 11, (McGraw-Hill, 1997).

[12] A. Altland, B. D. Simons, M. R. Zirnbauer, Physics Reports **359**, 283 (2002); M. J. Bhaseen, J. -S. Caux, I. I. Kogan, A. M. Tsvelik, Nuclear Physics **B618**, 465 (2001).